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Stephen Hawking's 1966 Adams Prize Essay

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The Adams Prize is awarded each year by the Faculty of Mathematics at the University of Cambridge and St John's College to a young, UK based mathematician for first-class international research in the Mathematical Sciences[1]. The Prize is named after the mathematician John Couch Adams. It was endowed by members of St John's College and was approved by the senate of the university in 1848 to commemorate Adams' discovery of the planet Neptune. Each year applications are invited from mathematicians who have worked in a specific area of mathematics. The prize has been awarded to many well known mathematicians and physicists, including James Clerk Maxwell (1857), J.J. Thomson (1882), John H. Poynting (1893), Joseph Larmor (1899), Geoffrey I. Taylor (1915), James H. Jeans (1917), Ralph H. Fowler (1924), Harold Jeffreys (1926), Abram S. Besicovitch (1930), William V.D. Hodge (1936), Subrahmanyan Chandrasekhar (1948), George Batchelor (1950), and Abdus Salam (1958).

In 1966 the topic set was "Geometric Problems of Relativity, with special reference to the foundations of general relativity and cosmology", with adjudicators Hermann Bondi, William V.D. Hodge, and A. Geoffrey Walker[2]. Roger Penrose submitted an essay entitled "An analysis of the structure of spacetime" and Stephen Hawking one called "Singularities and the Geometry of Spacetime". Penrose was awarded the Adams prize for his essay, and Hawking was awarded an auxiliary Adams Prize at the same time. Hawking's essay, reprinted below [Hawking 2014], summarised his work on global properties of general relativity theory, and in particular developed a series of cosmological singularity theorems he had proved. Neither Adams Prize essay was published as a book at the time, although preprint versions of both were circulated in the relativity community.

The context was the growing investigation of global properties of solutions of the general relativity field equations at that time. Two remarkable developments started this trend: first, Kurt Gödel's proof of the existence of exact solutions of the field equations for ordinary matter that allowed causality violation [Gödel 1949].

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[1] I am indebted to Wikipedia for the material in this paragraph; see http://www.maths.cam.ac.uk/news/4.html for the present day Adams Prize webpage, which gives no historical information.

[2] I thank Gary Gibbons for locating this information in the St John's College Archives.

Second, the recently obtained understanding of the unexpected global properties of the maximally extended Schwarzschild vacuum solution [Kruskal 1960]. Additionally the study of causal properties of spacetimes was making interesting headway [Zeeman 1964; Penrose 1963].

The issue of the existence of spacetime singularities was a vexing one, as emphasized particularly by John Wheeler [Wheeler 1963], because it meant an end to the predictability of physics. It was known that there were spacetime singularities (i) in the maximally extended Schwarzschild vacuum solution [Kruskal 1960], (ii) in the case of spherically symmetric gravitational collapse of a star [Oppenheimer and Snyder 1939; Wheeler 1963], and (iii) at the start of the standard Friedmann-Lemaître-Robertson-Walker (FLRW) cosmological models [Bondi 1960]. But these are all exact solutions with high symmetry. The issue was whether singularities would still exist in more realistic solutions, where for example rotation might spin up to allow singularity avoidance.

Many tried to investigate these issues by examining details of the Einstein field equations and their solutions, but the outcome was indeterminate. The issue was leaving the coordinate "singularity" concept in favour of a more physical coordinate free concept of a singularity. The main avenue used at that time in the cosmological case was looking at exact and approximate solutions of the field equations to see what they could tell us. At first we all tried to show a scalar invariant quantity would generically diverge, but without success. Particularly, following the introduction of Bianchi spatially homogeneous cosmologies by Gödel [Gödel 1952] (Bianchi IX matter solutions), Taub [Taub 1951] (generic vacuum cases), and Heckman and Schücking [Heckmann and Schücking 1962], Larry Shepley in Princeton, myself in Cambridge and others were trying to find singularity free fluid filled Bianchi cosmological models. It was obvious they would be singular in the orthogonal case when the rotation vanished, because of the Raychaudhuri equation ([Raychaudhuri 1955] for the pressure-free (geodesic) case, beautifully extended to generic fluids by Juergen Ehlers [Ehlers 1961]). The issue was what happened in the tilted case, when rotation might occur and prevent a singularity; but the analysis was indeterminate because one had to solve all 10 Einstein equations.

Penrose's highly innovative 1965 paper [Penrose 1965] showing existence of singularities in the generic context of a star collapsing to form a black hole revolutionised all this. Instead of focussing on the details of the field equations, Penrose used a new approach to showing the existence of singularities at the endpoint of gravitational collapse, based on global methods, the study of causal structure, and energy inequalities.

Three key concepts in this revolutionary paper were,

- The new definition of a singularity given by Penrose – the occurrence of an inextendible incomplete timelike or null geodesic must indicate existence of a spacetime singularity. That is, if there was a curve $x^a(v)$ with tangent vector L^a, $L_a L^a \leq 0$, $L_{a;b}L^b = 0$ such that the affine parameter v could only exist for finite values in the future, then the possible history of a particle moving on that path comes to an end; thus physics breaks down at that point. This definition opened up the way to general singularity theorems.
- The idea of a closed trapped 2-surface, that is a spacelike 2-sphere where both the ingoing and outgoing orthogonal future null geodesics are converging. This occurs in the maximally extended Schwarzschild vacuum solution as well as in the vacuum domain surrounding a collapsing star once it has passed inside its Schwarzschild radius. They will also occur in perturbed such solutions because they are defined by an inequality: both the null convergence $\theta_+(v) < 0$ and $\theta_-(v) < 0$.
- The energy conditions for physically reasonable matter, such as $\rho + p/c^2 \geq 0$, which will be true for all ordinary matter. On using the Null Raychaudhuri

equation – developed from the timelike case by Ehlers and Sachs [Jordan, Ehlers, and Sachs 1961], and which was also a particular one of the Newman-Penrose equations [Newman and Penrose 1962] – this ensured that conjugate points (places where null geodesics starting off from the same origin self-intersected) occurred along timelike or null geodesics, if closed trapped surfaces occurred.

In conjunction with a careful analysis of the boundaries of causal domains, this showed that when such closed trapped surfaces exist in realistic gravitational collapse contexts, this necessarily leads to the existence of incomplete timelike geodesics in inextendible spacetimes: that is, there would be an end to the possible histories of particles. Thus the spacetime must be singular [Penrose 1965; Hawking and Ellis 1973, 1965].

After the publication of this paper in January 1965, the members of Dennis Sciama's general relativity group in the Department of Applied Mathematics and Theoretical Physics at Cambridge University (particularly Stephen Hawking, myself, and Brandon Carter) hurriedly tried to learn the new methods that Penrose had introduced. We were assisted in this by discussions with Felix Pirani and the group at King's College, London; with John Wheeler and Charles Misner, who visited Cambridge from the USA for an extended period; and with Roger Penrose and Bob Geroch, who was visiting Penrose at Birkbeck College, London. In particular we had a one day seminar in Cambridge attended by the members of the King's college group, where I and Brandon Carter summarized our understandings of the ingredients of Penrose's theorem. Part of the requisite material was the coordinate-free approach to differential geometry that I had introduced to the Cambridge group in lectures in late 1964 after learning it from the excellent book by Helgason [Helgason 1962] and Cohn [Cohn 1957]; part was the material on focusing of timelike and null geodesics that we had learnt from Schücking, Ehlers, and Sachs [Ehlers 1961; Jordan, Ehlers, and Sachs 1961] and from Newman and Penrose [Newman and Penrose 1962]; part was the study of causal structure we had heard in seminars given by Penrose [Penrose 1963]; and part was the new material introduced by Penrose on how the focussing of null geodesics at caustics and folds bounded causal domains [Penrose 1965].

A key event was the GR4 International Conference on Relativistic Theories of Gravitation held at Imperial College, London, from 1–10 July 1965. Part of the official report on this event [GRG Bulletin 1965] was as follows:

> *The international 1965 conference on general relativity and gravitation was held at the Imperial College in London, July 1–9, and was organized by H. Bondi (King's College, London). The conference consisted of lectures, reviews, and several seminars including J.M. Khalatnikov (Moscow): On the singularities of the cosmological solutions of the gravitational equations. He made an attempt to give an answer to the question, whether the general solution of the gravitational equations has a singularity. The study led to the general conclusion that the presence of a singularity with respect to time is not a necessary feature of cosmological models; the general case of an arbitrary distribution of matter and field does not lead necessarily to the appearance of a singularity. This result, however, does not exclude the existence of a more restricted class of cosmological solutions, which possess a true singularity.*

This refocused our attention on the issue of the cosmological singularity. Hoyle was arguing against singularities at this meeting[3], because of his adherence to the Steady State universe idea. Khalatnikov's paper challenged us to find out what the answer was.

Stephen and I first focused on trying to prove existence of singularities in spatially homogeneous (Bianchi) world models, using these new methods. This led to the

[3] See [Wheeler 1996], p. 59.

idea of a Cauchy Horizon when the surfaces of transitivity of the group of symmetries became null, which was the difficult case to consider; but we managed to show this case was singular [Hawking and Ellis 1965]. Stephen then went on to consider generic inhomogeneous expanding cosmologies, which led to his key insight that time reversed closed trapped surfaces would occur in this case. On this basis he generalised to the cosmological case Penrose's global approach to showing the existence of singularities at the endpoint of gravitational collapse [Penrose 1965], and wrote a series of papers giving various theorems applicable to the cosmological case (some with errors in them). I presented some of these results on his behalf at an international meeting on gravitation in Miami Beach in 1965[4]. The results of three of these papers [Hawking 1965a, 1966a, 1966b] as well as his 1965 Ph.D. Thesis [Hawking 1965b] were included in the 1966 Adams Prize essay, as well as results from his innovative paper on covariant perturbations of a Robertson-Walker universe [Hawking 1966c]. The essay used results on causality by Kronheimer and Penrose [Kronheimer and Penrose 1967] and by Carter (unpublished at that stage). Stephen arrived at those results by discussions with the Cambridge group that under Dennis Sciama's guidance met to discuss ideas at tea time each day, and with the London groups (as well as attending many seminars, we used to regularly catch the train to attend lectures on general relativity at King's College, London).

Results then came apace. Later Hawking papers considering singularity theorems under somewhat different incomplete conditions and with different causality relations followed [Hawking 1966d, 1967], and papers on cosmic time functions [Hawking 1968], conservation of matter [Hawking 1970], and stable properties of solutions [Hawking 1971], developing from work by Carter on causality conditions [Carter 1971]. Geroch proved a further singularity theorem [Geroch 1966] and properties of the domain of dependence [Geroch 1970]. A paper by Hawking with myself showed that the mere existence of the cosmic background radiation, discovered in 1965, would ensure existence of closed trapped surfaces and hence the existence of singularities [Hawking and Ellis 1968], see also [Hawking and Ellis 1973]. Various papers investigated alternative definitions of a singularity [Geroch 1968; Kundt 1968; Schmidt 1971]. Hawking and Penrose jointly developed a general singularity theorem unifying the previous results [Hawking and Penrose 1970], whose abstract reads as follows:

> *A new theorem on space-time singularities is presented which largely incorporates and generalizes the previously known results. The theorem implies that space-time singularities are to be expected if either the universe is spatially closed or there is an 'object' undergoing relativistic gravitational collapse (existence of a trapped surface) or there is a point p whose past null cone encounters sufficient matter that the divergence of the null rays through p changes sign somewhere to the past of p (i.e. there is a minimum apparent solid angle, as viewed from p for small objects of given size). The theorem applies if the following four physical assumptions are made: (i) Einstein's equations hold (with zero or negative cosmological constant), (ii) the energy density is nowhere less than minus each principal pressure nor less than minus the sum of the three principal pressures (the 'energy condition'), (iii) there are no closed timelike curves, (iv) every timelike or null geodesic enters a region where the curvature is not specially alined with the geodesic. (This last condition would hold in any sufficiently general physically realistic model.) In common with earlier results, timelike or null geodesic incompleteness is used here as the indication of the*

[4] Stephen was having trouble speaking at that time, and so could not present it on his own – this was long before he had a computer speech synthesizer. I was based at Austin, Texas, at that time, and flew with Jane and Stephen from Austin to give this talk.

> *presence of space-time singularities. No assumption concerning existence of a global Cauchy hypersurface is required for the present theorem.*

All this and extra material on exact solutions, existence and uniqueness theorems, and the occurrence and nature of black holes were presented in *The Large Scale Structure of Space Time* [Hawking and Ellis 1973], which gave a more extensive discussion of the topics in the essay, as well as including some new topics. A book by Penrose summarised much of this work at more or less the same time [Penrose 1972]. Further work on existence and nature of singularities followed [Tipler 1977, 1978; Ellis and Schmidt 1977] in particular trying to understand the nature of the singularities predicted by the theorems (which only demonstrated geodesic incompleteness). Developments to 1980 are summarised in [Tipler, Clarke, and Ellis 1980].

The overall result was convincing evidence that classical general relativity theory implied a singular start to the universe. However of course one expects that quantum gravity will take over at very high densities so the real implication is that we have to contemplate a quantum gravity era in the very early universe. Thus the theorems provide an impetus to consider the nature of quantum gravity, and what it implies for the start of the universe – an ongoing project.

As to the essay itself: according to the Abstract, the aim of the essay is to investigate certain aspects of the geometry of the spacetime manifold in the General Theory of Relativity with particular reference to the occurrence of singularities in cosmological solutions and their relation with other global properties.

Section 2 gives a brief outline of Riemannian geometry, using coordinate free methods.

In Section 3, the General Theory of Relativity is presented in the form of two postulates and two requirements which are common to it and to the Special Theory of Relativity, and a third requirement, the Einstein field equations, which distinguish it from the Special Theory. "*There does not seem to be any alternative set of field equations which would not have some undesirable features*" – but many alternatives are now under active investigation, particularly due to the late time acceleration of the universe [Dodelson 2003; Ellis, Maartens, and MacCallum 2012], possibly due to "dark energy", and the dynamic indications of existence of dark matter, both of which could be due rather to the validity of other gravitational equations than the Einstein equations. Some exact solutions are given, including the Friedmann-Robertson-Walker solutions. Hawking states

> "*The singularity is the most striking feature of the Robertson-Walker solutions. It would imply that the universe (or at least that part of which we can have physical knowledge) had a beginning a finite time ago. However, it must be emphasized that this result depended on assuming exact spatial homogeneity and spherical symmetry.*"

That is the motivation for the later work.

In Section 4, the physical significance of curvature is investigated using the deviation equation for timelike and null curves, which follows on the work of Synge [Synge 1934], Pirani [Pirani 1956, 1957], Sachs, Penrose, and others. The deviation equation is standard but the matrix method used to analyse it is the author's own. The Riemann tensor is decomposed into the Ricci tensor which represents the gravitational effect at a point of matter at that point, and the Weyl tensor (the "free gravitational field") which represents the effect at a point of gravitational radiation and matter at other points, a viewpoint that has since been developed further in many later studies [Kristian and Sachs 1966; Ellis 1971]. The two tensors are related by the Bianchi identities, which, following work of Kundt and Trümper [Kundt and Trümper 1962], are presented in a form analogous to Maxwell's equations for electromagnetism (see [Maartens and Bassett 1998] for a more recent presentation). Some lemmas are given for the occurrence of conjugate points on timelike and null geodesics and their relation with the variation of timelike and null curves is established; this is an extension of Morse theory for positive definite spaces. Variation of curves and conjugate points are standard in a positive-definite metric but this seems to be the first full account for timelike and null curves in a Lorentz metric.

Section 5 is concerned with properties of causal relations between points of spacetime, following the work of Zeeman [Zeeman 1964] and Kronheimer and Penrose [Kronheimer and Penrose 1967]. It is shown that these could be used to determine physically the manifold structure of spacetime if the strong causality assumption held. The concepts of a null horizon and a partial Cauchy surface are introduced and are used to prove a number of lemmas relating to the existence of a timelike curve of maximum length between two sets. The nature of particle and event horizons is discussed, following the work of Rindler [Rindler 1956] and Penrose [Penrose 1963].

In Section 6, following Penrose, the definition of a singularity of spacetime is given in terms of geodesic incompleteness. The various energy assumptions needed to prove the occurrence of singularities are discussed and then a number of theorems are presented which prove the occurrence of singularities in most cosmological solutions, with the timelike and null versions of Raychaudhuri's equation [Raychaudhuri 1955] playing a key role. This is the essential new work in the essay: it extended Penrose's arguments to the cosmological case. A procedure is given which could be used to describe and classify the singularities and their expected nature is discussed. An Appendix discusses asymptotic behaviour near a singularity.

Overall, Hawking states in his essay,

> "*Undoubtedly, the most important results are the theorems in Section 6 on the occurrence of singularities. These seem to imply either that the General Theory of Relativity breaks down or that there could be particles whose histories did not exist before (or after) a certain time. The author's own opinion is that the theory probably does break down, but only when quantum gravitational effects become important. This would not be expected to happen until the radius of curvature of spacetime became about* 10^{-14} *cm.*"

That assessment stands. However apart from the issue of quantum gravity, it is now generally accepted, following on the success of Alan Guth's concept of an early inflationary epoch in the universe's history [Guth 1981], that scalar fields may occur which violate the timelike convergence condition (this possibility was already mentioned in [Hawking and Ellis 1973]), so invalidating one of the key assumptions going into the singularity theorems once quantum field theories become relevant. One can even find possible universes where a quantum gravity region never occurs: a bounce can be made possible by a suitable effective scalar field, or the universe could start from an initial static state that is balanced by an effective cosmological constant at very early times [Ellis and Maartens 2004]. There are additionally many quantum gravity options: pre-big bang scenarios and cyclic universes of various kinds. None of them however are based in well-tested physics.

Hence the issue of whether a spacetime singularity occurs in the real universe remains open. We do not in fact know if the universe had a beginning or not. Arguments have been given both ways: the issue remains undecided. Nevertheless the work in the essay is an important step forward: a clear statement of the fundamentally important implication of the classical theory that in the cosmological context, spacetime singularities will indeed exist. This sets the context within which quantum gravity studies of the start of the universe must be located.

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